1 Ex. 1

Consider a spin 1/2 particle. No external magnetic field is applied. Three measurements are done one after the other. In the first one the z component of the angular momentum is measured, in the second one the component along the direction \( \hat{u} \) is measured and in the third measurement, again the z component is measured. The unit vector \( \hat{u} \) is described using the angles \( \theta \) and \( \varphi \)
\[ \hat{u} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta). \]
(1)

Calculate the probability \( p_{\text{same}} \) to have the same result in the 1st and 3rd measurements.

1.1 Solution

The eigenvectors of \( S \cdot \hat{u} \) with eigenvalues \( \pm \hbar/2 \) are given by
\[ |+; S \cdot \hat{u}> = \cos \frac{\theta}{2} e^{-i \frac{\varphi}{2}} |+\rangle + \sin \frac{\theta}{2} e^{i \frac{\varphi}{2}} |–\rangle, \]
(2a)
\[ |–; S \cdot \hat{u}> = – \sin \frac{\theta}{2} e^{-i \frac{\varphi}{2}} |+\rangle + \cos \frac{\theta}{2} e^{i \frac{\varphi}{2}} |–\rangle, \]
(2b)
where the states \(|\pm\rangle\) are eigenvectors of \( S \cdot \hat{z} \). Let \( P(\sigma_3, \sigma_2 | \sigma_1) \) be the probability to measure \( S \cdot \hat{u} = \sigma_2 (\hbar/2) \) in the second measurement and to measure \( S \cdot \hat{z} = \sigma_3 (\hbar/2) \) in the third measurement given that the result of the first measurement was \( S \cdot \hat{z} = \sigma_1 (\hbar/2) \), and where \( \sigma_n \in \{+, –\} \). The following holds
\[ P(+, +|+) = |\langle +| +; S \cdot \hat{u} |\rangle|^2 |\langle +| +; S \cdot \hat{u} |\rangle|^2 = \cos^4 \frac{\theta}{2}, \]
(3a)
\[ P(+, –|+) = |\langle +| –; S \cdot \hat{u} |\rangle|^2 |\langle +| –; S \cdot \hat{u} |\rangle|^2 = \sin^4 \frac{\theta}{2}, \]
(3b)
\[ P(–, –|–) = |\langle –| –; S \cdot \hat{u} |\rangle|^2 |\langle –| –; S \cdot \hat{u} |\rangle|^2 = \cos^4 \frac{\theta}{2}, \]
(3c)
\[ P(–, +|+) = |\langle –| +; S \cdot \hat{u} |\rangle|^2 |\langle –| +; S \cdot \hat{u} |\rangle|^2 = \sin^4 \frac{\theta}{2}, \]
(3d)
thus independently on what was the result of the first measurement one has
\[ p_{\text{same}} = \cos^4 \frac{\theta}{2} + \sin^4 \frac{\theta}{2} = 1 – \frac{1}{2} \sin^2 \theta. \]
(4)

2 Ex. 2

Consider an Hydrogen atom. A perturbation given by
\[ V = Ar, \]
(5)
where \( r = \sqrt{x^2 + y^2 + z^2} \) is the radial coordinate and \( A \) is a constant is added.
1. Calculate to first order in $A$ the energy of the level ground state.
2. Calculate to first order in $A$ the energy of the first excited state.

### 2.1 Solution

The wavefunctions for the unperturbed case are given by

\[ \psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_{m}^{l}(\theta, \phi), \]  

where for the states relevant to this problem

\begin{align*}
R_{10}(r) &= 2 \left( \frac{1}{a_{0}} \right)^{3/2} e^{-r/a_{0}}, \\
R_{20}(r) &= (2 - r/a_{0}) \left( \frac{1}{2a_{0}} \right)^{3/2} e^{-\frac{r}{a_{0}}} , \\
R_{21}(r) &= \left( \frac{1}{2a_{0}} \right)^{3/2} \frac{r}{\sqrt{3a_{0}}} e^{-\frac{r}{2a_{0}}} , \\
Y_{0}^{0}(\theta, \phi) &= \sqrt{\frac{1}{4\pi}} , \\
Y_{-1}^{1}(\theta, \phi) &= \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{-i\phi} , \\
Y_{1}^{0}(\theta, \phi) &= \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta , \\
Y_{1}^{1}(\theta, \phi) &= -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{i\phi} ,
\end{align*}

and the corresponding eigenenergies are given by

\[ E_{n}^{(0)} = -\frac{E_{l}}{n^{2}} , \]  

where

\[ E_{l} = \frac{m_{e}e^{4}}{2\hbar^{2}}. \]  

The perturbation term $V$ in the Hamiltonian is given by $V = Ar$. The matrix elements of $V$ are expressed as

\[ \langle n'|l'm'|V|nlm \rangle = A \int_{0}^{\infty} dr \int_{-1}^{1} d(\cos \theta) \int_{0}^{2\pi} d\phi \left( Y_{l}^{m}^{*} \right)^{*} Y_{l}^{m} \]  

\[ = A \delta_{l,l'} \delta_{m,m'} \int_{0}^{\infty} dr \int_{-1}^{1} d(\cos \theta) \int_{0}^{2\pi} d\phi \left( Y_{l}^{m}^{*} \right)^{*} Y_{l}^{m} . \]  

\[ \]
1. Thus, to first order

\[ E_1 = E_1^{(0)} + \langle 100 | V | 100 \rangle + O(A^2), \]

where

\[ \langle 100 | V | 100 \rangle = A \int_0^\infty dr \, r^3 R_{10}^2 (r) = \frac{3Aa_0}{2}. \]

2. The first excited state is degenerate, however, as can be seen from Eq. (10) all off-diagonal elements are zero. The diagonal elements are given by

\[ \langle 200 | V | 200 \rangle = A \int_0^\infty dr \, r^3 R_{20}^2 = 6Aa_0, \]  
\[ \langle 21m | V | 21m \rangle = A \int_0^\infty dr \, r^3 R_{21} = 5Aa_0. \]

Thus, the degeneracy is lifted

\[ E_{2,l=0} = E_2^{(0)} + 6Aa_0 + O(A^2), \]
\[ E_{2,l=1} = E_2^{(0)} + 5Aa_0 + O(A^2). \]

3 Ex. 3

A rigid rotator is prepared in a state

\[ |\alpha\rangle = A (|1, 1\rangle - |1, -1\rangle), \]

where \( A \) is a normalization constant, and where the symbol \(|l, m\rangle\) denotes an angular momentum state with quantum numbers \( l \) and \( m \). Calculate

1. \( \langle L_x \rangle \)

2. \( \left\langle (\Delta L_x)^2 \right\rangle \)

3.1 Solution

The normalization constant can be chosen to be \( A = 1/\sqrt{2} \). The following holds:

\[ L_x = \frac{L_+ + L_-}{2}, \]
\[ L_+ |l, m\rangle = \hbar \sqrt{l(l+1) - m(m+1)} |l, m+1\rangle, \]
\[ L_- |l, m\rangle = \hbar \sqrt{l(l+1) - m(m-1)} |l, m-1\rangle. \]
1. The following holds

\[ L_x |\alpha\rangle = \frac{(L_{-1,1} - L_{+1,-1})}{2\sqrt{2}} \]

\[ = \frac{\hbar (|1,0| - |1,0|)}{2} = 0, \]  

(20)

thus

\[ \langle L_x \rangle = 0 . \]  

(21)

2. Using \( L_x |\alpha\rangle = 0 \) one finds

\[ \langle (\Delta L_x)^2 \rangle = \langle L_x^2 \rangle - \langle L_x \rangle^2 = 0 - 0 = 0 . \]  

(22)

4 Ex. 4

Consider a particle having mass \( m \) in a 3D potential given by

\[ V (r) = -A \delta (r - a) , \]

where \( r = \sqrt{x^2 + y^2 + z^2} \) is the radial coordinate, the length \( a \) is a constant and \( \delta () \) is the delta function. For what range of values of the constant \( A \) the particle has a bound state.

4.1 Solution

The radial equation is given by

\[ \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{l (l+1) \hbar^2}{2mr^2} + V (r) \right] u_{k,l} (r) = E_{k,l} u_{k,l} (r) . \]  

(23)

The boundary conditions imposed upon \( u (r) \) by the potential are

\[ u (0) = 0 , \]  

\[ u (a^+) = u (a^-) \]  

(24)

\[ \frac{du (a^+)}{dr} - \frac{du (a^-)}{dr} = -\frac{2}{a_0} u (a) . \]  

(25)

(26)

where

\[ a_0 = \frac{\hbar^2}{mA} . \]  

(27)

Since the centrifugal term \( l (l+1) \hbar^2/2mr^2 \) is non-negative the ground state is obtained with \( l = 0 \). We seek a solution for that case having the form

\[ u (r) = \begin{cases} 
\sinh (\kappa r) & r < a \\
\sinh (\kappa a) \exp (-\kappa (r - a)) & r > a
\end{cases} , \]  

(28)

4
where
\[ \kappa = \sqrt{-\frac{2mE}{\hbar}} . \]  

(29)

The condition (26) yields
\[ -\kappa \sinh (\kappa a) - \kappa \cosh (\kappa a) = -\frac{2}{a_0} \sinh (\kappa a) , \]  

or
\[ \frac{\kappa a_0}{2} = \frac{1}{1 + \coth (\kappa a)} . \]

A real solution exists only if
\[ \frac{a_0}{2} < a , \]  

or
\[ A > \frac{\hbar^2}{2ma} . \]  

(32)

5 Ex. 5

The Hamiltonian of a system is given by
\[ H = \epsilon N , \]  

(33)

where the real non-negative parameter \( \epsilon \) has units of energy, and where the operator \( N \) is given by
\[ N = b^\dagger b . \]  

(34)

The following holds
\[ b^\dagger b + bb^\dagger = 1 , \]  

(35)

\[ b^2 = 0 , \]  

(36)

\[ (b^\dagger)^2 = 0 . \]  

(37)

1. Find the eigenvalues of \( H \). Clue: show first that \( N^2 = N \).

2. Let \( |0\rangle \) be the ground state of the system, which is assumed to be non-degenerate. Define the two states
\[ |+\rangle = A_+ (1 + b^\dagger) |0\rangle , \]  

(38a)

\[ |-\rangle = A_- (1 - b^\dagger) |0\rangle , \]  

(38b)

where the real non-negative numbers \( A_+ \) and \( A_- \) are normalization constants. Calculate \( A_+ \) and \( A_- \). Clue: show first that \( b^\dagger |0\rangle \) is an eigenvector of \( N \).

3. At time \( t = 0 \) the system is in the state
\[ |\alpha (t = 0)\rangle = |+\rangle , \]  

(39)

Calculate the probability \( p (t) \) to find the system in the state \( |-\rangle \) at time \( t > 0 \).
5.1 Solution

The proof of the clue is:

\[ N^2 = b^\dagger b b^\dagger b = b^\dagger (1 - b^\dagger b) b = N. \]  

(40)

1. Let \(|n\rangle\) be the eigenvectors of \(N\) and \(n\) the corresponding real eigenvalues (\(N\) is Hermitian)

\[ N |n\rangle = n |n\rangle. \]  

(41)

Using the clue one finds that \(n^2 = n\), thus the possible values of \(n\) are 0 (ground state) and 1 (excited state). Thus, the eigenvalues of \(H\) are 0 and \(\varepsilon\).

2. To verify the statement in the clue we calculate

\[ Nb^\dagger |0\rangle = b^\dagger b b^\dagger |0\rangle = b^\dagger (1 - N) |0\rangle = b^\dagger |0\rangle, \]  

(42)

due to the fact that \(b^\dagger |0\rangle\) is an eigenvector of \(N\) with eigenvalue 1 (excited state). In what follows we use the notation

\[ |1\rangle = b^\dagger |0\rangle. \]  

(43)

Note that \(|1\rangle\) is normalized since

\[ \langle 1|1\rangle = \langle 0| b^\dagger b |0\rangle = \langle 0| (1 - N) |0\rangle = \langle 0|0\rangle = 1. \]  

(44)

Moreover, since \(|0\rangle\) and \(|1\rangle\) are eigenvectors of an Hermitian operator with different eigenvalues they must be orthogonal to each other

\[ \langle 0|1\rangle = 0. \]  

(45)

Using Eqs. (43), (44) and (45) one finds

\[ \langle +|+\rangle = 2 |A_+|^2, \]  

(46a)

\[ \langle -|\rangle = 2 |A_-|^2. \]  

(46b)

choosing the normalization constants to be non-negative reals lead to

\[ A_+ = A_- = \frac{1}{\sqrt{2}}. \]  

(47)

3. Using \(N^2 = N\) one finds

\[
\exp \left( -\frac{iHt}{\hbar} \right) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{i\varepsilon t}{\hbar} \right)^n \\
= 1 + N \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{i\varepsilon t}{\hbar} \right)^n \\
= 1 + N \left( -1 + \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{i\varepsilon t}{\hbar} \right)^n \right) \\
= 1 + N \left( -1 + \exp \left( -\frac{i\varepsilon t}{\hbar} \right) \right). \]

(48)
Thus

\[ p_0(t) = \left| -\exp\left( \frac{-iHt}{\hbar} \right) + \right|^2 \]

\[ = \frac{1}{4} \left| \langle 0 \rangle - \langle 1 \rangle \right| \left[ 1 + N \left( -1 + \exp\left( \frac{-iDt}{\hbar} \right) \right) \right] \left| \langle 0 \rangle + \langle 1 \rangle \right|^2 \]

\[ = \frac{1}{4} \left| 1 - \exp\left( \frac{-iDt}{\hbar} \right) \right|^2 \]

\[ = \sin^2 \left( \frac{et}{2\hbar} \right). \]

(49)