3.10 Wedge in a Waveguide

3.10.1 Metallic Wedge
The goal of this section is to analyze a wave guide of a circular cross-section but which lacks a slice as illustrated below.

In order to keep the problem relatively simple, we shall limit the discussion to the evaluation of the cut-off frequency. For this purpose we recall that in Section 2.1 we
found that

$$\tilde{E}_\perp = -\frac{jk_z}{k_\perp^2} \nabla_\perp E_z + \frac{j\omega\mu}{k_\perp^2} \mathbf{1}_z \times \nabla_\perp H_z$$

$$\tilde{H}_\perp = -\frac{jk_z}{k_\perp^2} \nabla_\perp H_z - \frac{j\omega\varepsilon}{k_\perp^2} \mathbf{1}_z \times \nabla_\perp E_z$$

$$k_\perp^2 = \omega^2 \varepsilon \mu - k^2$$

(3.10.1)

And both longitudinal components satisfy the wave equation. Limiting the analysis to the cut-off frequencies $k_z = 0$ these equations simplify

$$\tilde{E}_\perp = -\frac{1}{j\omega\varepsilon} \mathbf{1}_z \times \nabla_\perp H_z \Rightarrow \begin{cases} E_r = \frac{1}{j\omega\varepsilon} \frac{1}{r} \frac{\partial}{\partial \phi} H_z \\ E_\phi = -\frac{1}{j\omega\varepsilon} \frac{\partial}{\partial r} H_z \end{cases}$$

$$\tilde{H}_\perp = +\frac{1}{j\omega\mu} \mathbf{1}_z \times \nabla_\perp E_z \Rightarrow \begin{cases} H_r = -\frac{1}{j\omega\mu} \frac{1}{r} \frac{\partial}{\partial \phi} E_z \\ H_\phi = \frac{1}{j\omega\mu} \frac{\partial}{\partial r} E_z \end{cases}$$

(3.10.2)

and the wave equation simplifies
\[
\left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \omega^2 \mu \epsilon \right) \begin{pmatrix} E_z \\ H_z \end{pmatrix} = 0
\] (3.10.3)

Obviously, the solution of this set of equations describe a resonator whereby the field does not vary in the z-direction.

For the **TM modes** the solution is of the form

\[
E_z(r, \phi) \propto J_v(qr) \left[ A \cos(v\phi) + B \sin(v\phi) \right]
\] (3.10.4)

subject to the boundary conditions

\[
E_z(r = R, \alpha / 2 < \phi < 2\pi - \alpha / 2) = 0
\]

\[
E_z(r, \phi = \alpha / 2) = 0
\]

\[
E_z(r, \phi = 2\pi - \alpha / 2) = 0
\] (3.10.5)

The first condition implies that

\[
J_v(qR) = 0
\] (3.10.6)

wherein \( q \) is yet to be determined and from the other two conditions we will specify \( v \):

\[
E_z(r, \phi = \alpha / 2) = 0 \Rightarrow A \cos(v\alpha / 2) + B \sin(v\alpha / 2) = 0
\]

\[
E_z(r, \phi = 2\pi - \alpha / 2) = 0 \Rightarrow A \cos(v(2\pi - \alpha / 2)) + B \sin(v(2\pi - \alpha / 2)) = 0
\] (3.10.7)

or explicitly
For a non-trivial solution the determinant of the matrix is zero

\[
\begin{vmatrix}
\cos\left(\nu \alpha / 2\right) & \sin\left(\nu \alpha / 2\right) \\
\cos\left[\nu (2\pi - \alpha / 2)\right] & \sin\left[\nu (2\pi - \alpha / 2)\right]
\end{vmatrix} = 0
\]

(3.10.8)

For a non-trivial solution the determinant of the matrix is zero

\[
\nu = \frac{\pi n}{2\pi - \alpha} = n \frac{1}{2 - \alpha / \pi}
\]

(3.10.9)

wherein \( n = 1, 2, 3, \ldots \infty \). Now that we have determined \( \nu \) we may proceed and establish the zeros of the Bessel function:

\[
J_{\frac{1}{2 - \alpha / \pi}}(p) = 0
\]

(3.10.10)

As an example let us consider three Bessel functions in one case \( \nu = 1 \) and for comparison we illustrate below \( \alpha = \pi / 5 \) thus \( \nu(n = 1) = 5/9 \) and \( \nu(n = 2) = 10/9 \).

Obviously the Bessel function of a non-integer order has also zeros:

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( s = 1 )</th>
<th>( s = 2 )</th>
<th>( s = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu = 1 )</td>
<td>3.832</td>
<td>7.016</td>
<td>10.173</td>
</tr>
<tr>
<td>( \nu(n = 1) = 5/9 )</td>
<td>3.220</td>
<td>6.366</td>
<td>9.509</td>
</tr>
<tr>
<td>( \nu(n = 2) = 10/9 )</td>
<td>3.981</td>
<td>7.175</td>
<td>10.337</td>
</tr>
</tbody>
</table>
Explicitly

\[ E_z (r, \phi) = \sum_{n,s=1}^{\infty} U_{n,s} J_{n-1,2-\alpha/\pi} \left( p_{n,s} \frac{r}{R} \right) \sin \left( \frac{\pi n \phi - \alpha / 2}{2\pi - \alpha} \right) \]

and the cut-off frequencies are determined by

\[ \omega_{co} = \frac{c}{R} p_{n,s} \]

The top frame in the right reveals the contours of constant \( E_z \) for \( s = 1 \) and \( n = 1 \). The central frame illustrates the case \( s = 1 \) and \( n = 2 \) whereas the lower frame corresponds to \( s = 2 \) and \( n = 1 \).
A similar approach may be followed for the TE modes

\[ H_z(r, \phi) \propto J_v(qr) \left[ A \cos(v\phi) + B \sin(v\phi) \right] \quad (3.10.13) \]

subject to the boundary conditions

\[ E_\phi(r = R, \alpha / 2 < \phi < 2\pi - \alpha / 2) = 0 \]
\[ E_r(r, \phi = \alpha / 2) = 0 \quad (3.10.14) \]
\[ E_r(r, \phi = 2\pi - \alpha / 2) = 0 \]

The first condition [Eq. (3.10.2)] implies that

\[ \left[ \frac{d}{dr} J_v(qr) \right]_{r=R} = 0 \quad (3.10.15) \]

wherein \( q \) is yet to be determined and from the other two conditions we specify \( v \):

\[ E_r(r, \phi = \alpha / 2) = 0 \Rightarrow -A \sin(v\alpha / 2) + B \cos(v\alpha / 2) = 0 \]
\[ E_r(r, \phi = 2\pi - \alpha / 2) = 0 \Rightarrow -A \sin[v(2\pi - \alpha / 2)] + B \cos[v(2\pi - \alpha / 2)] = 0 \quad (3.10.16) \]

or explicitly

\[ \begin{pmatrix} -\sin(v\alpha / 2) & \cos(v\alpha / 2) \\ -\sin[v(2\pi - \alpha / 2)] & \cos[v(2\pi - \alpha / 2)] \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \quad (3.10.17) \]

For a non-trivial solution the determinant of the matrix is zero
\[ \nu = \frac{\pi n}{2\pi - \alpha} = n \frac{1}{2 - \alpha \pi} \]  
\hspace{1cm} (3.10.18)  

wherein \( n = 1, 2, 3, \ldots \infty \). Now that we have determined \( \nu \) we may proceed and establish the zeros of the Bessel function:

\[ J'_{n \frac{1}{2 - \alpha / \pi}} (p) = 0 \]  
\hspace{1cm} (3.10.19)

From here the approach is virtually identical

\[ H_z (r, \phi) = \sum_{n,s=1}^{\infty} U_{n,s} J_{n \frac{1}{2 - \alpha / \pi}} \left( p'_{n,s} \frac{r}{R} \right) \cos \left( \frac{\pi n \phi - \alpha / 2}{2\pi - \alpha} \right) \]  
\hspace{1cm} (3.10.20)

\[ p'_{n,s} : J'_{n \frac{1}{2 - \alpha / \pi}} (p) = 0 \]

**Exercise 3.22:** Draw the contours of constant \( H_z \). Compare the field distribution of the TE and TM modes. For both modes compare the magnetic and electric energy per-unit length.
3.10.2 Dielectric Wedge

The goal of this section is to analyze a waveguide of a circular cross-section but which has a material \((\mu_r, \varepsilon_r)\) slice as illustrated below.

As in the metallic case we keep the problem relatively simple, and limit the discussion to the evaluation of the cut-off frequency. As a result, we can separate the analysis of the TE and TM modes. So let us start with the TM mode.

\[
\begin{align*}
\text{Vacuum:} & \quad \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\omega^2}{c^2} \right) E_z = 0 \\
H_r &= -\frac{1}{j\omega\mu_0} \frac{1}{r} \frac{\partial}{\partial \phi} E_z \\
H_\phi &= \frac{1}{j\omega\mu_0} \frac{\partial}{\partial r} E_z \\
\text{Slice:} & \quad \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\omega^2}{c^2} \mu_r\varepsilon_r \right) E_z = 0 \\
H_r &= -\frac{1}{j\omega\mu_0\mu_r} \frac{1}{r} \frac{\partial}{\partial \phi} E_z \\
H_\phi &= \frac{1}{j\omega\mu_0\mu_r} \frac{\partial}{\partial r} E_z
\end{align*}
\]

A differential approach (separation of variables) is not applicable since the radial...
variation needs to be the same in both regions but if this is the case the function does not satisfy the wave equation. We must seek an integral approach. For determining the cut-off frequencies associated with the TM-like modes we need to solve

\[
\left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\omega^2}{c^2} \right) E_z = -\frac{\omega^2}{c^2} f(\phi) E_z
\]  

(3.10.21)

wherein \( f(\phi) \) is a function which equals \( \mu_r \varepsilon_r - 1 \) in the region(s) with material and zero otherwise. A solution which satisfies the boundary conditions

\[
E_z(r,\phi) = \sum_{n=-\infty}^{\infty} \sum_{s=1}^{\infty} \mathcal{E}_{n,s} J_n \left( \frac{p_{n,s}}{R} \right) \exp(jn\phi)
\]  

(3.10.22)

substituting in (3.10.21) we get

\[
\mathcal{E}_{n,s} \chi_{n,n} \left( -\frac{p_{n,s}^2}{R^2} + \frac{\omega^2}{c^2} \right) = -\frac{\omega^2}{c^2} \sum_{m=-\infty}^{\infty} \sum_{\sigma=1}^{\infty} \mathcal{E}_{m,\sigma} \chi_{m,n}^{\sigma,s} F_{n',m}
\]

\[
\chi_{n,n}^{s,s'} \equiv \int_0^1 dx J_{n'} \left( p_{n',s'} x \right) J_n \left( p_{n,s} x \right), \quad \chi_{n,n}^{s,s'} = \delta_{s,s'} \int_0^1 dx J_{n'} \left( p_{n,s} x \right) J_n \left( p_{n,s} x \right)
\]  

(3.10.23)

\[
F_{n',m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi f(\phi) \exp[-j(n'-m)\phi]
\]
Rewriting
\[
\left( \frac{\omega}{c} R \right)^2 \sum_{m=-\infty}^{\infty} \sum_{\sigma=1}^{\infty} \mathcal{E}_{m,\sigma} \chi_{m,n}^{\sigma,s} \left[ F_{n,m} + \delta_{n,m} \delta_{\sigma,s} \right] = \mathcal{E}_{n,s} \chi_{n,n}^{s,s} p_{n,s}^2 \quad (3.10.24)
\]
and defining the matrices
\[
C_{\{n,s\},\{m,\sigma\}} = \frac{\chi_{m,n}^{\sigma,s}}{\chi_{n,n}^{s,s} p_{n,s}^2} \left[ F_{n,m} + \delta_{n,m} \delta_{\sigma,s} \right] \quad (3.10.25)
\]
thus defining $\bar{\omega} = \omega R / c$.

\[
\sum_{\{m,\sigma\}} C_{\{n,s\},\{m,\sigma\}} \mathcal{E}_{\{m,\sigma\}} = \bar{\omega}^{-2} \sum_{\{m,\sigma\}} I_{\{n,s\},\{m,\sigma\}} \mathcal{E}_{\{m,\sigma\}} \quad (3.10.26)
\]
implying that the (square of the) cut-off frequencies are the inverse of the eigen-values of the matrix $C$

\[
|C - \bar{\omega}^{-2} I| = 0. \quad (3.10.27)
\]

**Exercise 3.23:** Calculate the first cut-off frequency of a waveguide with dielectric slice identical with the metallic slice. Compare the two results. Examine the convergence of the solution.

**Exercise 3.24:** Repeat Exercise 3.23 for the TE mode including the formulation.