

Theorem 1. Let \( N_t \sim \text{Exp} (\mu) \) be independent and identically distributed random variables, and let \( N_i \sim \text{Exp} (\lambda) \) be independent and identically distributed random variables for each \( i \). Then,

\[
\mathbb{E} [ N_t - N_s | N_s ] = \mathbb{E} [ N_t - N_s ] = \mathbb{E} [ \mathbb{E} [ N_t - N_s | \lambda ] ] = \mathbb{E} [ \lambda (t - s) ] = \frac{t - s}{\mu}
\]

1. Assume that the two random variables are independent and identically distributed.

2. Let \( \lambda = 1/\mu \) and \( \mathbb{E} [ N_i ] = 1/\mu \).

Theorem 2. Let \( N_t \sim \text{Exp} (\mu) \) be independent and identically distributed random variables, and let \( X_i \) be independent and identically distributed random variables. Then,

\[
\mathbb{E} X_t = \sum_{i=1}^{N_t} X_i, \quad \text{Var} X_t = \sum_{i=1}^{N_t} \text{Var} X_i
\]

1. Mean and variance of the sum of independent random variables.

2. The expected value and variance of the sum of independent random variables.

3. The expected value and variance of the sum of independent random variables.
The vector $X_t, t \geq 0$ is a homogeneous Poisson process. We consider the case $X_t, t \geq 0$.

**Solution**

1. **Mean**

$$E[X_t] = E \left[ \sum_{i=1}^{N_t} B_i \middle| N_t \right] = E[N_t \cdot p] = \lambda t$$

2. **Variance**

$$\text{Var}[X_t] = E[X_t^2] - (E[X_t])^2$$

$$= E \left[ \sum_{i=1}^{N_t} \sum_{j=1}^{N_t} B_i B_j \middle| N_t \right] - (\lambda t)^2$$

$$= E \left[ (N_t^2 - N_t)p^2 + N_t \cdot p \right] - (\lambda t)^2$$

$$= \lambda tp^2 + (\lambda t)^2 - \lambda tp^2 + \lambda t - (\lambda t)^2$$

$$= \lambda tp$$

3. **Cross-covariance.** Given $t_1 < t_2$, we have

$$R_X(t_1, t_2) = E[X_{t_1} X_{t_2}]$$

$$= E \left[ \sum_{i=1}^{N_{t_1}} \sum_{j=1}^{N_{t_2}} B_i B_j \middle| N_{t_1}, N_{t_2} \right]$$

$$= E \left[ \sum_{i=1}^{N_{t_1}} \sum_{j=1}^{N_{t_2}} B_i B_j + \sum_{i=1}^{N_{t_1}} \sum_{j=N_{t_1}+1}^{N_{t_2}} B_i B_j \middle| N_{t_1}, N_{t_2} \right]$$

$$= E \left[ N_{t_1} \cdot p + (N_{t_1}^2 - N_{t_1})p^2 + N_{t_1} (N_{t_2} - N_{t_1})p^2 \right]$$

$$= \lambda t_1 p + \lambda t_1 p^2 + (\lambda t_1)^2 p^2 - \lambda t_1 p^2 + \lambda t_1 (\lambda t_2 - \lambda t_1)p^2$$

$$= \lambda t_1 p + \lambda^2 t_1 t_2 p^2$$

In summary,

$$R_X(t_1, t_2) = \lambda p \min(t_1, t_2) + \lambda^2 p^2 t_1 t_2$$

4. **Initial Condition**

- If $E(0) = 0$.
- To verify the properties of $X_t$, we define

$$h(i, n) = \mathbb{P} \left\{ \sum_{k=1}^{n} B_k = i \right\}$$
i.i.d. $B_i$. Let $\{X_t\}$ be a Poisson process with rate $\lambda$, for all $t \geq 0$.

$$
P \{ X_t = i, X_s - X_t = j \} = \mathbb{E} \left[ \mathbb{E} \left[ \left. X_t = i, X_s - X_t = j \right| N_t, N_s \right] \right]
$$

$$
= \mathbb{E} \left[ \mathbb{P} \left( \sum_{k=1}^{N_t} B_k = i, \sum_{l=N_t}^{N_s} B_l = j \right| N_t, N_s \right) \right]
$$

$$
= \mathbb{E} \left[ \mathbb{E} \left( \left. h(i, N_t) h(j, N_s - N_t) \right| N_t \right) \right]
$$

$$
= \mathbb{E} \left[ h(i, N_t) \right] \cdot \mathbb{E} \left[ h(j, N_s - N_t) \right]
$$

$$
P \{ X_t = i \} \cdot P \{ X_s - X_t = j \}
$$

\[ \text{Cov}(N_t, N_s) = \mathbb{E}(N_t N_s) - \mathbb{E}(N_t) \mathbb{E}(N_s) \]

\[ \rho_{N_t, N_s} = \frac{\text{Cov}(N_t, N_s)}{\sqrt{\text{Var}(N_t) \text{Var}(N_s)}} \]

\( R_y(\tau), R_x(\tau) \) are the time-varying covariance and correlation, respectively, of $y_t$ and $x_t$.

\( R_x(\tau) \) and $R_y(\tau)$ can be defined as

$$
R_x(\tau) = \mathbb{E} [x_{t+\tau}^2]
$$

$$
R_y(\tau) = \mathbb{E} [y_{t+\tau}^2]
$$

\[ R_{xy}(\tau) = \mathbb{E} [x_{t+\tau} y_{t+\tau}] \]

\[ R_{xz}(\tau) = \mathbb{E} [x_{t+\tau} z_{t+\tau}] \]

\[ R_{yz}(\tau) = \mathbb{E} [y_{t+\tau} z_{t+\tau}] \]
\[ R_x(\tau) = E[zt_{z+\tau}] \]
\[ = E[p_r y_r p_{t+\tau} y_{t+\tau}] \]
\[ = E[p_r y_r] E[p_{t+\tau} y_{t+\tau}] \]
\[ + E[p_r y_{t+\tau}] E[y_r p_{t+\tau}] \]
\[ + E[p_r p_{t+\tau}] E[y_r y_{t+\tau}] \]
\[ = R_y^2(0) + R_y^2(\tau) R_x(\tau) + R_y(\tau) R_x(\tau) \]

\[ R_y(\tau) = R_y(-\tau) \]
\[ = E[y_r(x + h)_{t+\tau}] \]
\[ = E\left[y_r \int_{-\infty}^{\infty} x_{\hat{h}_{t+\tau-\theta}} d\theta \right] \]
\[ = \int_{-\infty}^{\infty} E[y_r x_{\theta}] h_{t+\tau-\theta} d\theta \]
\[ = \int_{-\infty}^{\infty} R_{yx}(\theta - t) h_{t+\tau-\theta} d\theta \]
\[ = (R_{yx} * h)(\tau) \]

\[ E[Y_1(t)Y_2(t+\tau) = E\left[\int_{-\infty}^{\infty} X_1(\alpha) h_1(t - \alpha) d\alpha \int_{-\infty}^{\infty} X_2(\beta) h_2(t + \tau - \beta) d\beta \right] \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{X_1,X_2}(\beta - \alpha) h_1(t) h_2(t + \tau) d\alpha d\beta \]
\[ \gamma_{\hat{\beta} = \alpha}^{\beta} \int_{-\infty}^{\infty} h_1(t - \alpha) d\alpha \int_{-\infty}^{\infty} R_{X_1,X_2}(\gamma) h_2(t + \tau - \alpha - \gamma) d\gamma \]
\[ \theta_{\hat{t} = -\alpha}^{\tau} \int_{-\infty}^{\infty} h_1(\theta) (R_{X_1,X_2} * h_2)(\theta + \tau) d\theta \]
\[ = R_{X_1,X_2}(u) * h_2(u) * h_1(-u) \]
\[ S_{Y_1,Y_2}(f) = S_{X_1,X_2}(f)H_1^*(f)H_2(f) \]

\[ \begin{aligned}
X_t & \xrightarrow{H_1(f)} Y_t \\
& \xrightarrow{H_2(f)} \hat{X}_t
\end{aligned} \]

Given the mutual information of the set \( \{X_t\} \) and \( \{N_t\} \) with \( H_2 \) of the LTIs \( S_N(f) \) is calculated.


e \left[ X^2Y^2 \right] = 2 \left( E[XY] \right)^2 + E \left[ X^2 \right] E \left[ Y^2 \right]

For the case of the system to operate properly, for \( X_0 \) and \( Y_0 \), the following conditions must be met:

1. The variables \( \{X_t\} \) and \( \{N_t\} \) are independent.
2. The transfer function \( H_1(f) = j2\pi f \) is valid for all frequencies.
3. The minimum value of the transfer function is \( H_1(f) = 1 \) for all frequencies.
4. The mutual information is \( I(X_t;Y_t) \) for all frequencies.
5. The mutual information is \( I(X_t;\hat{X}_t) \) for all frequencies.

Moreover, for the system to be effective, the following conditions must be met:

1. The parameters \( \{\hat{X}_t\} \) of the LTIs are fixed.
2. The output \( \{\hat{X}_t\} \) is obtained from the input \( \{X_t\} \) of the LTIs.
3. The system is effective for all frequencies.

\[ E \left[ \hat{X}_t^2 \right] = R_X(0) \]


\[ E \left[ X_t^2 X_{t+\tau}^2 \right] = 2 \left( E \left[ X_t X_{t+\tau} \right] \right)^2 + E \left[ X_t^2 \right] E \left[ X_{t+\tau}^2 \right] = 2 \left( R_X(\tau) \right)^2 + \left( R_X(0) \right)^2 \]

The above integrals are evaluated under the assumptions of the model.

\[ \text{Cov}(X_t, Y_t) = E \left[ \frac{d}{dt} X_t + N_t \right] = E \left[ \frac{d}{dt} X_t \right] + E [N_t] = \frac{d}{dt} E \left[ X_t^2 \right] = 0 \]

The covariance is zero for all \( t \neq 0 \).

\[ E \left[ \epsilon^2 \right] = \int_{-\infty}^{\infty} S_X(f) \left[ 1 - H_1(f) H_2(f) \right]^2 + S_N(f) |H_2(f)|^2 \, df \]

The expression above is the mean square error of the model.

\[ H_2(f) = \frac{S_X(f) |H_1|^2}{S_X(f) |H_1|^2 + S_N(f)} \]

The above expression is the transfer function of the model.

\[ E \left[ \epsilon^2 \right] = \int_{-\infty}^{\infty} \frac{S_X(f) S_N(f) + S_X^2(f) S_N(f) |H_1(f)|^2}{S_X(f) |H_1(f)|^2 + S_N(f)} \, df = \int_{-\infty}^{\infty} S_X(f) S_N(f) \, df \]

The above expression is the mean square error of the model.
Theorem 1: Equation 6

For a system in which the input $X_t$ and the output $Y_t$ are defined by the graph:

$$
H_t = \begin{cases} 1 & \text{if } t \leq W \\ -1 & \text{if } t > W 
\end{cases}
$$

and $H_t$ is a white noise process, the output $Y_t$ can be expressed as:

$$
Y_t = X_t + W_t
$$

where $W_t$ is Gaussian white noise.

Theorem 2: Equation 9

For a system in which the input $X_t$ and the output $Y_t$ are defined by the graph:

$$
H_t = \begin{cases} 1 & \text{if } t \leq W \\ -1 & \text{if } t > W 
\end{cases}
$$

and $H_t$ is a white noise process, the output $Y_t$ can be expressed as:

$$
Y_t = X_t + W_t
$$

where $W_t$ is Gaussian white noise.

Theorem 3: Equation 10

For a system in which the input $X_t$ and the output $Y_t$ are defined by the graph:

$$
H_t = \begin{cases} 1 & \text{if } t \leq W \\ -1 & \text{if } t > W 
\end{cases}
$$

and $H_t$ is a white noise process, the output $Y_t$ can be expressed as:

$$
Y_t = X_t + W_t
$$

where $W_t$ is Gaussian white noise.

Theorem 4: Equation 11

For a system in which the input $X_t$ and the output $Y_t$ are defined by the graph:

$$
H_t = \begin{cases} 1 & \text{if } t \leq W \\ -1 & \text{if } t > W 
\end{cases}
$$

and $H_t$ is a white noise process, the output $Y_t$ can be expressed as:

$$
Y_t = X_t + W_t
$$

where $W_t$ is Gaussian white noise.

Theorem 5: Equation 12

For a system in which the input $X_t$ and the output $Y_t$ are defined by the graph:

$$
H_t = \begin{cases} 1 & \text{if } t \leq W \\ -1 & \text{if } t > W 
\end{cases}
$$

and $H_t$ is a white noise process, the output $Y_t$ can be expressed as:

$$
Y_t = X_t + W_t
$$

where $W_t$ is Gaussian white noise.

Theorem 6: Equation 13

For a system in which the input $X_t$ and the output $Y_t$ are defined by the graph:

$$
H_t = \begin{cases} 1 & \text{if } t \leq W \\ -1 & \text{if } t > W 
\end{cases}
$$

and $H_t$ is a white noise process, the output $Y_t$ can be expressed as:

$$
Y_t = X_t + W_t
$$

where $W_t$ is Gaussian white noise.

Theorem 7: Equation 14

For a system in which the input $X_t$ and the output $Y_t$ are defined by the graph:

$$
H_t = \begin{cases} 1 & \text{if } t \leq W \\ -1 & \text{if } t > W 
\end{cases}
$$

and $H_t$ is a white noise process, the output $Y_t$ can be expressed as:

$$
Y_t = X_t + W_t
$$

where $W_t$ is Gaussian white noise.
הسكنטרוימר של \( Y \):

\[
S_Y(f) = S_Z(f) + S_n(f)
\]

\[
S_Y(f) = S_X(f) |H_1(f)|^2 + N_0
\]

\[
= N_0 \mathbb{1}_{(|f| \leq 1)} + N_0
\]

בSİ_N טבל

\[
H_{opt}(f) = \frac{1}{2}, \quad |f| \leq 1
\]

\[
0, \quad \text{אחרת}
\]

\[
S_Y(f) = S_Z(f) + S_n(f)
\]

\[
Z_t \rightarrow \text{עקב אי תחרהט ב} \, X_t
\]

\[
\text{נ IPV תיחלור אט הסכמטור הצ שY}
\]

\[
S_Z(f) = \mathcal{F} \{ \mathbb{E} [Z_t Z_{t+1}] \}
\]

\[
= \mathcal{F} \{ \mathbb{E} [\mathbb{E} [Z_t Z_{t+1} \mid W]] \}
\]

\[
= \mathbb{E}_W [\mathcal{F} \{ \mathbb{E} [Z_t Z_{t+1} \mid W] \}]
\]

\[
= \mathbb{E}_W \left[ N_0 \mathbb{1}_{(|f| \leq W)} \right]
\]

\[
= N_0 (1 - |f|^3) \mathbb{1}_{(|f| \leq 1)}
\]

כシャרי אט תחרהטל ב-(??) תישבע טאפ נבט:

\[
\mathbb{E}_W \left[ \mathbb{1}_{(|f| \leq W)} \right] = \mathbb{P} \{-W < f < W\}
\]

\[
= \mathbb{P} \{ W > |f| \}
\]

\[
= \int_{|f|}^{1} f_W(\alpha) d\alpha
\]

\[
= 1 - |f|^3
\]

לקט

\[
S_Y(f) = N_0 + N_0 (1 - |f|^3) \mathbb{1}_{(|f| \leq 1)}
\]

_amp7

\[
\text{א лично או גוות ביאור}
\]

\[
f_Z(\alpha) = \int_{0}^{1} f_{Z \mid W}(\alpha|w) f_W(w) dw = \int_{0}^{1} \frac{1}{\sqrt{4\pi w N_0}} e^{-\alpha^2/4wN_0} 3w^2 dw
\]

האיטטיטר לאinic טמכותא טאן ליק� אט תחרהטל באטס.

\[
\text{أمرダン האיטטיטר טאן ליקי}
\]

\[
H_{opt}(f) = \frac{S_{Y \cdot X}(f)}{S_Y(f)}
\]
$S_Y(f)$

At the spectrum of $S_Y(f)$

$$S_Y(f) = \mathcal{F}\{\mathbb{E}[Y_t X_{t+\tau}]\}$$

$$= \mathcal{F}\{\mathbb{E}[Z_t X_{t+\tau}]\}$$

$$= \mathcal{F}\{\mathbb{E}[E[Z_t X_{t+\tau}|W]]\}$$

$$= \mathbb{E}_W[\mathcal{F}\{E[Z_t X_{t+\tau}|W]\}]$$

$$= \mathbb{E}_W[N_0 \mathbb{I}_{(|f|\leq W)}]$$

$$= N_0 (1 - |f|^3) \mathbb{I}_{(|f|\leq 1)}$$

$H_{\text{opt}}(f) = \begin{cases} 
\frac{1-|f|^3}{2-|f|^3}, & |f| \leq 1 \\
0, & \text{otherwise}
\end{cases}$

MLE of $\alpha$:

$$0 = \epsilon_1 \leq \epsilon_2 \leq \epsilon_3$$

Horizontality test - Null hypothesis of Horizontality

The null hypothesis $H_0$ is that the spectrum of $X_t$ is zero in the frequency domain $\mathbb{E}[X_t X_{t+\tau}] = 0$, $\forall \tau$. The alternative hypothesis $H_1$ is that the spectrum is nonzero in the frequency domain $\mathbb{E}[X_t X_{t+\tau}] \neq 0$, $\forall \tau$.

$$\alpha \geq 0$$

$$\mathbb{P}\{\Delta_i \leq 0\} = 0, \alpha < 0$$

$$\mathbb{P}\{\Delta_i \leq \alpha\} = 1 - \mathbb{P}\{t_i - t_{i-1} > \alpha\} = 1 - \mathbb{E}[\mathbb{P}\{t_i - t_{i-1} > \alpha | t_{i-1}\}]$$

$$\mathbb{P}\{t_i - t_{i-1} > \alpha | t_{i-1} = \tau\} = \mathbb{P}\{\text{no events in } (\alpha + \tau, \tau) | t_{i-1} = \tau\}$$

$$\mathbb{P}\{\text{no events in } (\alpha + \tau, \tau)\} = \exp\{-\lambda \alpha\}$$
\[ F_{\Delta_i}(\alpha) = \begin{cases} 1 - \exp\{-\lambda \alpha\}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases} \]

\[ \Delta_i \sim \text{Exp}(\lambda) \]

calculated

Theorem 8 (Someone, Equations) for the continuous case, its modified form and application. The modified form is given by

\[ N(t) = \int_0^t \lambda e^{-\lambda s} ds \]

Theorem 1: The modified form is given by

\[ X_1(t) = \int_0^t X(t) \, dt \]

and for \( X_2(t) \):

\[ \Delta_i \sim \text{Exp}(\lambda) \]

Theorem 1 and 2: The modified form is given by

\[ \Delta_i \sim \text{Exp}(\lambda) \]

Theorem 3: The modified form is given by

\[ X_1(t) = \int_0^t X(t) \, dt \]

and for \( X_2(t) \):

\[ \Delta_i \sim \text{Exp}(\lambda) \]

Theorem 4: The modified form is given by

\[ X_1(t) = \int_0^t X(t) \, dt \]

and for \( X_2(t) \):

\[ \Delta_i \sim \text{Exp}(\lambda) \]

Theorem 5: The modified form is given by

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נניח כי ממד בוהלולים דטרמינסטיים ממדים את המשוואה הנתונה
\[ 2\pi j Y(f) = -aY(f) + X(f) \]
מכן,
\[ Y(f) = \frac{1}{\frac{1}{a} + 2\pi j f} X(f) \]
כלומר התנדוקים
\[ H(f) = \frac{1}{\frac{1}{a} + 2\pi j f} \]
נ가입 את התנדוקים אי-האורוגרפיים של \( \{V(t)\} \) \( \{X(t)\} \) \( \{Y(t)\} \) \( \{Z(t)\} \). 2
\[ \mathbb{E}[V(t) - \mathbb{E}[Y(t)Z(t)] = \mathbb{E}[Y(t)\mathbb{E}[Z(t) = 0 \]
ואז מתקבל שטרומפלר של \( \{Y(t)\} \) אי אפס כמשתנה מפריכי ליידי
\[ R_V(\tau) = \mathbb{E}[V(t)V(t + \tau)] = \mathbb{E}[Y(t)Z(t)Y(t + \tau)Z(t + \tau)] = R_Z(\tau)R_Y(\tau) \]
לפיכךيفינו: $S_V(f) = \mathcal{F}\{ R_Y(\tau) \}$

$R_Y(\tau) = \mathcal{F}\left\{ \frac{N_0}{a^2 + \omega^2} \right\} = \frac{N_0}{2a} e^{-a|\tau|}$

לפי מיון זה: $S_V(f) = \mathcal{F}\{ R_Y(\tau) \}$

פ栎 $F\{ R_Y(\tau) \}$

$= \mathcal{F}\left\{ \frac{N_0}{8a} e^{-a|\tau|} + \frac{N_0}{8a} e^{-2(\lambda+\alpha)|\tau|} \right\}$

$= \frac{N_0}{4} \left( \frac{1}{a^2 + 4\pi^2 f^2} + \frac{2\lambda+\alpha}{4a (2\lambda+\alpha)^2 + 4\pi^2 f^2} \right)$

$\mathbb{E}[V(t)Y(t+\tau)] = \mathbb{E}[Y(t)Z(t)] \quad [V(t) = Y(t)-Y(t)\text{ נקודת נקודת וצמדה מנסים]}$

$H_{\text{opt}}(\omega) = \frac{S_{VY}(f)}{S_V(f)} = \frac{\frac{N_0}{4} \frac{1}{a^2 + 4\pi^2 f^2} + \frac{N_0}{4a} \frac{2\lambda+\alpha}{(2\lambda+\alpha)^2 + 4\pi^2 f^2}}{\frac{1}{2a^2 + 4\pi^2 f^2} + \frac{1}{2a (2\lambda+\alpha)^2 + 4\pi^2 f^2}}$

$V(t) = 0$ ו$Z(t) = 0$ מתקיים $Z(t) = 0$ והקשר בין $V(t) = Y(t)$-ה י الجميع $\hat{Y}(t)$. מתקיים $\hat{Y}(t)$-ה הגרاء זה נקודה הגראה הגראה(domino) המסרת הגראה domino שטוחה של$\hat{Y}(t)$.

$\hat{Y}(t) = \begin{cases} V(t), & V(t) \neq 0 \\ (V * h_{\text{opt}})(t), & V(t) = 0 \end{cases}$
Theorem 10 (Lemma 2): The non-linear gate, $H$, is a linear function of the input variables $X_t$ and $Y_t$, given by $Y_t = H(X_t, Y_t) = X_t + Y_t$.

Proof: The proof is by induction on the number of layers. For a single-layer network, the function $H(X_t, Y_t) = X_t + Y_t$ is linear. Assume that the function is linear for a network with $k$ layers. Consider a network with $k+1$ layers. Let $H_k$ be the function for the network with $k$ layers, $H_{k+1}$ be the function for the network with $k+1$ layers, and $H_{k+2}$ be the function for the network with $k+2$ layers. Then, $H_{k+1}(X_t, Y_t) = H_k(X_t, H_{k+1}(X_t, Y_t)), H_{k+2}(X_t, Y_t) = H_k(X_t, H_{k+1}(X_t, Y_t)), Y_t = H_{k+2}(X_t, Y_t) = X_t + Y_t$.

For a two-layer network, $H(X_t, Y_t) = X_t + Y_t$ is linear.
{X_t} \text{ is a Markov process with state } F \text{ and transition matrix } A \text{ and the time to absorption is } T.  

1. The expected value of the squared error for a given time step is:

\[ E \left[ (X_t - \hat{X}_t)^2 \right] = \int_{-W}^{W} S_\epsilon(f) df = \int_{-W}^{W} \frac{N_1 N_0}{N_1 + N_0} df = \frac{2W N_0}{1 + \frac{N_0}{N_1}} \]

where the expectation is taken over all realizations of the random field with density \( f \). The optimal value of \( N_0 \) is given by:

\[ N_0^* = \frac{1}{2} \frac{1 + \frac{W^2}{2}}{1 + \frac{W^2}{4}} \]

2. The generalization error is defined as:

\[ MSE = E \left[ (X_t - \hat{X}_t)^2 \right] 
= E \left[ E \left[ (X_t - \hat{X}_t)^2 | S \right] \right] 
= E \left[ \int_{-\infty}^{\infty} S_X(f) \left[ 1 - H(f)H_S(f) \right]^2 + S_n(f) | H(f)H_S(f) \right]^2 df \right] \]

3. The optimal filter is given by:

\[ H_{opt}(f) = \frac{S_{Y'X}(f)}{S_Y(f)} \]

4. The optimal filter is defined as:

\[ H_{opt}(f) = \frac{S_X(f)}{S_X(f) + S_n(f)} \cdot \frac{pG^*(f) + (1 - p)F^*(f)}{p |G(f)|^2 + (1 - p) |F(f)|^2} \]
\[
\mathbb{E} \left[ (X_t - \hat{X}_t)^2 \right] = \int_{-\infty}^{\infty} \frac{S_X(f)S_Y(f) - |S_{XY}(f)|^2}{S_Y(f)} \, df \\
= \int_{-\infty}^{\infty} S_X(f) \, df - \int_{-\infty}^{\infty} |S_X(f)|^2 \sqrt{\mathbb{E} \left[ H_0^*(f) \right]^2} \, df \\
= 2WN_0 - \int_{-\infty}^{\infty} \frac{S_X^2(f)}{S_X(f) + S_n(f)} (pG^*(f) + (1-p)F^*(f)) \, df \\
= 2WN_0 - \int_{-W}^{W} \frac{1}{2N_0(1-p)} \, df \\
= WN_0(1+p)
\]

Given that the random variables $\{X_t\}$ are independent and identically distributed (i.i.d.), we have $\mathbb{E} \left[ (X_t - \hat{X}_t)^2 \right] = WN_0 \cdot p = 0$, so $\mathbb{E} \left[ (X_t - \hat{X}_t)^2 \right] = 2WN_0$. Therefore, the variance of the estimator $0 < p < 1$.

5. For the case of dependent samples, we need to consider the dependence structure of the samples. The samples are dependent if the samples are taken from the same population or if there is a correlation between the samples.

\[
H_{opt}(s) = \frac{S_X(f)}{S_X(f) + S_n(f)} \cdot \frac{1}{F(f)}
\]

The new estimator is given by

\[
H_{new}(f) = \frac{S_X(f)}{S_X(f) + S_n(f)} \cdot \frac{1}{H(f)} = \frac{S_X(f)}{S_X(f) + S_n(f)} \cdot \frac{1}{F(f)}
\]

3. The parameter $p \in \{0, 1\}$ is used to model the dependence structure of the samples.

4. A new estimator $H(f)$ is derived, which includes the old estimator $S_{XY}(f)$ and the new estimator $S_{Y^*}(f)$.

The diagram shows the model structure:

\[
\begin{align*}
&Z_t \quad \text{Input} \\
&\quad \quad \text{1} \\
&\quad \quad H_1 \\
&\quad \quad \quad \text{X_t} \\
&\quad \quad \quad \quad \text{Y_t} \\
&\quad \quad \quad \quad \quad H \\
&\quad \quad \quad \quad \quad \quad \text{X_t}
\end{align*}
\]

The model structure includes:

- Input $Z_t$
- Transformation $H_1$
- Output $X_t$
- Transformation $H$
- Output $\hat{X}_t$
\( f \geq W \) then \( H(f) = 0 \). 1. If \( f \) is between \( a \) and \( b \) then \( H(f) = \alpha \), where \( \alpha = \frac{1}{2} \) if \( |f| \leq |W| \) otherwise \( \alpha = \frac{1}{2} \).

\( H(f) \leq \alpha \) for \( |f| > W/2 \). 2. \( \alpha = c = d \). 3. \( H(f) \leq \alpha \) for \( |f| > W/2 \).

\[
H(f) = \begin{cases} 0 & |f| \leq W/2 \\ \alpha & |f| > W/2 \end{cases}
\]

\( W \) is the width, \( \alpha \) is the offset, \( c \) and \( d \) are cut-off points.

\[
W = \frac{1}{2} \quad \alpha = \frac{1}{2}
\]

\[ f_{W}(\alpha) = \lambda e^{-\lambda}, \quad \alpha > 0 \]

\[ EW^k = k!/\lambda^k \quad \text{where} \quad W \sim \text{Exp(\lambda)} \]

\[
\begin{align*}
X_0 &= \frac{W^2}{\lambda^2} \\
S_X(f) &= X_0^{-1/2} \quad \text{where} \quad X_0 = \frac{W^2}{\lambda^2}
\end{align*}
\]

\[ X_0 = \frac{W^2}{\lambda^2} \quad \text{mode} \]

\[
\begin{align*}
H(f) &= \min \left\{ \frac{1}{2}, \alpha \right\} \\
\lambda &= \frac{W}{2} \\
c &= \min \left\{ \frac{1}{2}, \alpha \right\}
\end{align*}
\]

\[ f_{W}(\alpha) = \lambda e^{-\lambda}, \quad \alpha > 0 \]

\[
\begin{align*}
X_0 &= \frac{W^2}{\lambda^2} \\
S_X(f) &= X_0^{-1/2} \quad \text{mode} \quad X_0 = \frac{W^2}{\lambda^2}
\end{align*}
\]

\[ X_0 = \frac{W^2}{\lambda^2} \quad \text{mode} \]

\[
\begin{align*}
H(f) &= \min \left\{ \frac{1}{2}, \alpha \right\} \\
\lambda &= \frac{W}{2} \\
c &= \min \left\{ \frac{1}{2}, \alpha \right\}
\end{align*}
\]
\[ W(X_0^2) = \hat{W} + \frac{\text{Cov}(W, X_0^2)}{\text{Var}X_0^2}(X_0^2 - \mathbb{E}X_0^2) \]

We have at the mean square derivative:

\[ \mathbb{E}W = 1/\lambda \]

\[ \mathbb{E}X_0^2 = \mathbb{E}[\mathbb{E}[X_0^4|W]] = \mathbb{E}[R_{X|W}(0)] = \mathbb{E}\left[ \int_{-\infty}^{\infty} S_{X|W}(f) \, df \right] = \mathbb{E}[2N_0W] = \frac{2N_0}{\lambda} \]

\[ \text{Var}X_0^2 = \mathbb{E}[X_0^4] - (\mathbb{E}X_0^2)^2 = \mathbb{E}[\mathbb{E}[X_0^4|W]] - 4\frac{N_0^2}{\lambda^2} = \mathbb{E}[3(2N_0W)^2] - 4\frac{N_0^2}{\lambda^2} = 12N_0^2 \frac{2}{\lambda^2} - 4\frac{N_0^2}{\lambda^2} = \frac{20N_0^2}{\lambda^2} \]

\[ \text{Cov}(W, X_0^2) = \mathbb{E}[WX_0^2] - \mathbb{E}W\mathbb{E}X_0^2 = \mathbb{E}[2N_0W^2] - \frac{1}{\lambda} \cdot \frac{2N_0}{\lambda} = \frac{2N_0}{\lambda^2} \]

Based on this, we have:

\[ \hat{W}(X_0^2) = \frac{4}{5\lambda} + \frac{X_0^2}{10N_0} \]

For the second method:

\[ \mathbb{E}X_t = \mathbb{E}[\mathbb{E}[X_t|W]] = \mathbb{E}[\mathbb{E}[Z_t \ H_t(0)]] = 0 \]
\[ R_X(t, t + \tau) = \mathbb{E}[X_t X_{t+\tau}] \]
\[ = \mathbb{E}[\mathbb{E}[X_t X_{t+\tau} | W]] \]
\[ = \mathbb{E}[(R_Z \ast h_1 \ast \hat{h}_1)(\tau) | W] \]

Theorem 3.1 (continued): For the case of a single-input single-output system, the system can be expressed as:

\[ S_X(f) = \mathcal{F} \{ \mathbb{E}[X_t X_{t+\tau}] \} \]
\[ = \mathcal{F} \{ \mathbb{E}[\mathbb{E}[X_t X_{t+\tau} | W]] \} \]
\[ = \mathbb{E}_W \{ \mathcal{F} \{ \mathbb{E}[X_t X_{t+\tau} | W] \} \} \]
\[ = \mathbb{E}_W \{ N_0 \mathbb{1}_{|f| \leq W} \} \]
\[ = N_0 \mathbb{P} \{ W > |f| \} \]
\[ = N_0 \exp \{ -\lambda |f| \} \]

**Exercise 3.1 (a)** For the system \( \lambda = \frac{1}{\mathbb{E}[X_t X_{t+\tau}]} \) and system (\( \lambda = \frac{1}{\mathbb{E}[X_t X_{t+\tau}]} \):